Graphs having many holes but with small competition numbers

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Abstract

The *competition number* $k(G)$ of a graph G is the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph. A chordless cycle of length at least 4 of a graph is called a hole of the graph. The number of holes of a graph is closely related to its competition number as the competition number of a chordal graph which does not contain a hole is at most one and the competition number of a complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ which has so many holes that no more holes can be added is the largest among those of graphs with n vertices. In this paper, we show that even if a connected graph G has many holes, $k(G)$ can be as small as 2 under some assumption. In addition, we show that, for a connected graph G with exactly h holes and exactly one non-edge maximal clique, if all the holes of G are pairwise edge-disjoint and the size ω of the non-edge clique of G satisfies $3 \leq \omega \leq h+1$, then the competition number of G is at most $h - \omega + 3$.

Keywords: competition graph; competition number; hole; clique
2000 MSC: 05C75

1. Introduction

Let $D = (V, A)$ be a digraph (for all undefined graph-theoretical terms, see [1]). The *competition graph* $C(D)$ of D has the same vertex set as D and has an edge xy if for some vertex $v \in V$, the arcs (x, v) and (y, v) are in D. The notion of competition graph is due to Cohen [3] and has arisen from ecology. A *food web* in an ecosystem is a digraph whose vertices are the species of the system and which has an arc from a vertex u to a vertex v if and only if u preys on v. Given a food web F, it is said that species u and v compete if and only if they have a common prey. Competition graphs also have applications in coding, radio transmission, and modeling of complex economic systems. (See [10] and [12] for a summary of these applications and [4] for a sample paper on the modeling application.)

Roberts [11] observed that every graph together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. The *competition number* $k(G)$ of a graph G is defined to be the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph. That is, when I_k is a set of k isolated vertices, $k(G)$ is the smallest integer k such that the disjoint union $G \cup I_k$ is the competition graph of an acyclic digraph. It is well known that computing the competition number of a graph is an NP-hard problem [9]. It has been one of the important research problems in the study of competition graphs to characterize a graph by its competition number.

We call a cycle of a graph G a *chordless cycle* of G if it is an induced subgraph of G. A chordless cycle of length at least 4 of a graph is called a *hole* of the graph and a graph without holes is called a *chordal graph*. The number of holes of a graph is closely related to its competition number as the competition number of a chordal graph which does not contain a hole is at most one (see [11]) and the competition number of a complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ which has so many holes that no more holes can be added is the largest among those of graphs with n vertices (see [5]). In fact, the competition number of a triangle-free graph with only holes no two of which share an edge can be

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[1] The authors were supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2008-531-C00004).
[2] This work was supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0094069).

computed in terms of the number of its holes. Take a graph G such that G has exactly h holes and no two holes of G share an edge. Roberts [11] showed that if G is nontrivial, triangle-free and connected, then $k(G) = |E(G)| - |V(G)| + 2$. By this theorem, the competition number of G is $h+1$ as G has $h + |V(G)| - 1$ edges. Therefore $k(G)$ is almost as large as h. Then we naturally come up with an interesting question: "Is $k(G)$ still kept large if G is allowed to have just one maximal clique of size sufficiently large?". In this paper, we answer this question by showing that even if a connected graph G has many holes, $k(G)$ can be as small as 2 under some assumption. In addition, we show that, for a connected graph G with exactly h holes and exactly one non-edge maximal clique, if all the holes of G are pairwise edge-disjoint and the size ω of the non-edge clique of G satisfies $3 \leq \omega \leq h+1$, then the competition number of G is at most $h - \omega + 3$.

2. Main result

For a graph G and a set $S \subseteq V(G)$ of vertices of G, we denote by $G[S]$ the subgraph of G induced by S.

Lemma 2.1. *Let C be a cycle of length at least 4 in a graph G. If C has a chord, then the subgraph $G[V(C)]$ of G has a triangle or contains two holes which have a common edge.*

Proof. Let $C = v_1v_2v_3...v_n$ be a cycle of G and v_iv_j be a chord of C for some $i < j$. Two (v_i, v_j)-sections of C are (v_i, v_j)-walks of $G[\{v_i, v_{i+1}, ..., v_j\}] - v_iv_j$ and $G[\{v_j, v_{j+1}, ..., v_i\}] - v_iv_j$. Let P_1 and P_2 be shortest (v_i, v_j)-paths in $G[\{v_i, v_{i+1}, ..., v_j\}] - v_iv_j$ and $G[\{v_j, v_{j+1}, ..., v_i\}] - v_iv_j$, respectively. Since G is simple, the lengths of P_1 and P_2 are at least 2. If the length of P_1 or P_2 is 2, then $P_1 + v_iv_j$ or $P_2 + v_iv_j$ is a triangle in $G[V(C)]$. Otherwise, $P_1 + v_iv_j$ and $P_2 + v_iv_j$ are holes which have a common edge v_iv_j. □

A *clique* is a complete subgraph of a graph. A clique K is called *non-edge* if $|V(K)| \geq 3$.

Lemma 2.2. *Let G be a connected graph. Suppose that all the holes in G are pairwise edge-disjoint and that G has exactly one non-edge maximal clique K. Then, a cycle C in G is a holes if and only if it stisfies $|V(K) \cap V(C)| \leq 2$.*

Proof. The 'only if' part is obvious. We show the 'if' part by contradiction. Suppose that C is not a hole, that is, C has a chord. By Lemma 2.1, the subgraph $G[V(C)]$ of G has a triangle or contains two holes with a common edge. If $G[V(C)]$ has a triangle, then the triangle is a clique of size 3 different from K since $|V(K) \cap V(C)| \leq 2$, which is a contradiction. Otherwise, it contradicts the assumption that all the holes of G are edge-disjoint. Thus C is a hole. □

For a clique K in a graph G, we call a path P in G a *K-avoiding path* if P is not an edge of K and any of internal vertices of P is not on K.

Lemma 2.3. *Let G be a connected graph with exactly h holes. Suppose that all the holes in G are pairwise edge-disjoint and that G has exactly one non-edge maximal clique. If the non-edge maximal clique K in G has size $h+1$, then G contains a vertex $v \in K$ satisfying one of the following:*

(a) *there is no K-avoiding path from the vertex v to any vertex in any holes,*

(b) *the vertex v is incident to an edge common to K and a hole, and is not contained in any other hole.*

Proof. Let $H_1, H_2, ..., H_h$ be the holes of G. We define a bipartite multigraph B on bipartition (V_1, V_2), where $V_1 = V(K) = \{v_1, v_2, ..., v_{h+1}\}$ and $V_2 = \{H_1, H_2, ..., H_h\}$, as follows. Two vertices $v_i \in V_1$ and $H_j \in V_2$ are joined by r edges in B if there exists a K-avoiding path from v_i to a vertex in H_j, where r is defined by

$$r = \begin{cases} 2 & \text{if } v_i \text{ is a cut vertex in } G \text{ and any vertex in } V(K) \setminus \{v_i\} \text{ and any vertex in } V(H_j) \setminus \{v_i\} \\ & \text{belong to different components of } G - v_i, \\ 1 & \text{otherwise.} \end{cases}$$

If $\deg_B(v_i) = 0$ for some i, then v_i satisfies the condition (a). Suppose that $\deg_B(v_i) = 1$ for some i. Then there exists a unique j such that G has a K-avoiding path P from v_i to a vertex x in H_j. Therefore v_i is not contained in any other hole than H_j. Since $\deg_B(v_i) \neq 2$, G has a K-avoiding path

P' from $v_{i'} \in V(K) \setminus \{v_i\}$ to a vertex x' in H_j. Then the walk formed by $v_i v_{i'}$, P, a (x, x')-section of H_j, and P' contains a cycle. Then the edge $v_i v_{i'}$ is contained in a hole since G has exactly one non-edge maximal clique K. Thus v_i satisfies the condition (b). Hence what we have to prove is the following:

($*$) there exists $v_i \in V_1$ such that $\deg_B(v_i) \leq 1$.

To show the claim ($*$), we show that $\deg_B(H_j) \leq 2$ hold for all $1 \leq j \leq h$. Suppose that $\deg_B(H_j) \geq 3$ for some $j \in \{1, \ldots, h\}$. We will reach a contradiction.

First, we suppose that there are three distinct K-avoiding paths P_1, P_2, and P_3 going from the distinct vertices v_{i_1}, v_{i_2}, and v_{i_3} in K to vertices x_1, x_2, and x_3 in H_j, respectively. Since $V(H_j) \cap V(K) \leq 2$ by Lemma 2.2, without loss of generality, we may assume $v_{i_3} \notin V(H_j)$. Then the length of P_3 is at least 1. Let w be the vertex immediately following v_{i_3} on P_3. Then $w \notin V(K)$. If $v_{i_3}w$ is a cut edge of G, then any path from a vertex in K to a vertex in H_j must contain the edge $v_{i_3}w$. This implies that P_1 contains the vertex v_{i_3} as an internal vertex of P_1, which contradicts that P_1 is a K-avoiding path. Therefore $v_{i_3}w$ is not a cut edge, and so the edge $v_{i_3}w$ is contained in some cycle in G. Let C be a shortest cycle among the cycles containing the edge $v_{i_3}w$. By the choice of C, C has no chord. If C is a triangle, i.e., a clique of size 3, then C is a clique different from K since $w \notin V(K)$ and $w \in V(C)$, which is a contradiction. Thus C is a hole. Since $\{v_{i_1}, v_{i_2}, v_{i_3}\} \nsubseteq V(C)$ and $v_{i_3} \in V(C)$, $v_{i_1} \notin V(C)$ or $v_{i_2} \notin V(C)$. Without loss of generality, we may assume that $v_{i_1} \notin V(C)$. The (w, x_3)-section of P_3, an (x_3, x_1)-section of H_j and the (x_1, v_{i_1})-section of P_1 form a (w, v_{i_1})-walk W which does not contain v_{i_3}. Let Q be the shortest (w, v_{i_1})-path that is a subsequence of the (w, v_{i_1})-walk W. Then $C' = Qv_{i_3}w$ is a cycle. Here we note that $V(K) \cap V(C') = \{v_{i_1}, v_{i_3}\}$ by the definition. By Lemma 2.2, C' is a hole and we have reached a contradiction as $v_{i_3}w$ is an edge common to the holes C and C'.

Now suppose that $H_j \in V_2$ is incident to multiple edges. Let $v_{i_1} \in V_1$ be the other end of the multiple edges. Since $\deg_B(H_j) \geq 3$, there is another vertex v_{i_2} adjacent to H_j in B. By the definition of B, v_{i_1} is a cut vertex of G and no other vertex in K belongs to the component containing vertices of H_j in $G - v_{i_1}$. It contradicts to the existence of a K-avoiding path from v_{i_2} to a vertex in H_j which does not contain v_{i_1}.

Consequently, $\deg_B(H_j) \leq 2$ for all $1 \leq j \leq h$ and so

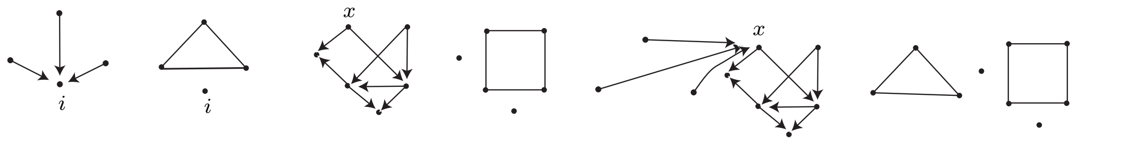
$$\sum_{i=1}^{h+1} \deg_B(v_i) = |E(B)| = \sum_{j=1}^{h} \deg_B(H_j) \leq 2h.$$

If $\deg_B(v_i) \geq 2$ for all $1 \leq i \leq h+1$, then $\sum_{i=1}^{h+1} \deg_B(v_i) \geq 2(h+1)$ and it is a contradiction. Therefore, there exists a vertex v_i with $\deg_B(v_i) \leq 1$ and so ($*$) holds. □

Lemma 2.4. *Let G be a connected graph with exactly h holes. Suppose that all the holes in G are pairwise edge-disjoint and that G has exactly one non-edge maximal clique K. If $G - e$ has at least h holes for some edge e of a hole H in G, then e is an edge of K. In particular, holes in $G - e$ but not in G have the form $(H - v_i v_j) \cup \{v_i v_k, v_j v_k\}$ where $e = v_i v_j$ and v_k is a vertex of K.*

Proof. To show it by contradiction, we suppose that $G - e$ has at least h holes for an edge $e = uv$ of a hole H which is not an edge of K. Since all the holes in G are edge-disjoint, any hole other than H does not contain the edge e. Since $G - e$ has at least h hole, e is a chord of a cycle distinct from H in G. That is, there exists a (u, v)-path P other than $H - e$. Without loss of generality, we may assume that P is a shortest path between u and v in $G - e$. Since G is simple, P is not an edge. If the length of P is 2, then $P + e$ is a triangle and so it is contained in K, which contradicts our assumption that e is not an edge of K. On the other hand, if the length of P is at least 3, then $P + e$ is a hole which is distinct from H. It is also a contradiction as e is an edge common to H and $P + e$. Therefore $G - e$ has at most $h - 1$ holes and it is also a contradiction. Consequently, e is an edge common to H and K. In addition, we can easily check that $H - e$ together edges $v_i v_k$ and $v_j v_k$ is a hole of $G - e$ where $e = v_i v_j$ and v_k is a vertex of K. □

Lemma 2.5. *Let D_1 and D_2 be acyclic digraphs such that $V(D_1) \cap V(D_2) = \emptyset$. Suppose that there are p vertices in D_1 which have no in-neighbors in D_1 and there are p isolated vertices in $C(D_2)$. Then there exists an acyclic digraph D such that $C(D) = C(D_1) \cup C(D_2) - I_p$, where I_p is a set of p isolated vertices in $C(D_2)$.*



D_1 $\quad C(D_1)$ $\quad D_2$ $\quad C(D_2)$ $\quad D$ $\quad C(D) = C(D_1) \cup C(D_2) - \{i\}$

Figure 1: D_1, D_2, and D.

Proof. Let u_1, u_2, $\dots$, u_p be vertices which have no in-neighbors in D_1 and $I_p = \{i_1, i_2, \dots, i_p\}$ be a set of p isolated vertices in $C(D_2)$. We define a digraph D with vertex set $V(D_1) \cup V(D_2) - I_p$ by changing the arcs incoming toward i_j to the arcs incoming toward u_j, that is,

$$A(D) = A(D_1) \cup A(D_2) - \bigcup_{j=1}^{p} \{(v, i_j) \mid v \in N^-_{D_1}(i_j)\} \cup \bigcup_{j=1}^{p} \{(v, u_j) \mid v \in N^-_{D_1}(i_j)\}$$

(see Figure 1 for an illustration). Then D is acyclic and $C(D) = C(D_1) \cup C(D_2) - I_p$. Hence the lemma holds. $\square$

In the following, we will prove the main theorem by induction. We prove the basis step first.

Lemma 2.6. *Let G be a connected graph with exactly two holes. Suppose that the holes in G are edge-disjoint and that G has exactly one non-edge maximal clique. If the non-edge maximal clique K has size three, then there is an acyclic digraph D such that $C(D) = G \cup \{i_1, i_2\}$ and all the vertices of K have a common out-neighbor i_1 or i_2.*

Proof. First we show that $k(G - E(K)) \leq 1$. Let $V(K) = \{v_1, v_2, v_3\}$. Note that the number of components of $G - E(K)$ is at most 3. We consider the following three cases.

Case 1: The number of the components of $G - E(K)$ is 1.

We show that $G - E(K)$ is a tree by contradiction. Suppose that $G - E(K)$ has a cycle C. Since $G - E(K)$ is connected in this case, there exist at least two of a (v_1, v_2)-path which does not contain v_3, a (v_2, v_3)-path which does not contain v_1, and a (v_3, v_1)-path which does not contain v_2 in $G - E(K)$. Without loss of generality, we may assume that there exist a (v_1, v_2)-path P_1 which does not contain v_3 and a (v_2, v_3)-path P_2 which does not contain v_1 in $G - E(K)$. Then, since P_1 is a K-avoiding (v_1, v_2)-path and P_2 is a K-avoiding (v_2, v_3)-path in $G - E(K)$, $C_1 := P_1 + v_1v_2$ and $C_2 := P_2 + v_2v_3$ are cycles in G other than C. Since $V(C_i) \cap V(K) = \{v_i, v_{i+1}\}$ for $i = 1, 2$, by Lemma 2.2, C_1 and C_2 are holes in G. Since C contains neither v_1v_2 nor v_2v_3, it is distinct from C_1 and C_2. Since there are exactly two holes, C cannot be a hole. Then C must have a chord with which two consecutive edges of C form a triangle. This triangle is different from K, which contradicts the hypothesis.

Case 2: The number of the components of $G - E(K)$ is 2.

Let G_1 and G_2 be the two components of $G - E(K)$. Since $V(K)$ is not contained in one component in $G - E(K)$, we may assume, without loss of generality, that $v_1, v_2 \in V(G_1)$ and $v_3 \in V(G_2)$. Then $\{v_1v_3, v_2v_3\}$ is an edge cut of G. Since v_1 and v_2 are in the same component, there is a (v_1, v_2)-path in G_1. Let P be a shortest (v_1, v_2)-path in G_1. Then, by Lemma 2.2, the cycle $C_1 := P + v_1v_2$ is a hole of G. Since $\{v_1v_3, v_2v_3\}$ is an edge cut, none of v_1v_3, v_2v_3 belongs to a hole. Thus $G - E(K)$ contains the other hole C_2 of G. Since C_2 is the only hole $G - E(K)$, either G_1 or G_2 is a tree. Without loss of generality, we may assume that G_1 is a tree. Then G_2 contains C_2. Since G_1 is a tree, $k(G_1) \leq 1$. Then there exists an acyclic digraph D_1 such that $C(D_1) = G_1 \cup \{i_1\}$ where i_1 is an isolated vertex. Note that D_1 contains two vertices x, y which have no in-neighbors. Since G_2 is connected, triangle-free and has exactly one hole, $k(G_2) = |E(G_2)| - |V(G_2)| + 2 = 2$. Then there exists an acyclic digraph D_2 such that $C(D_2) = G_2 \cup \{i_2, i_3\}$ where i_2 and i_3 are isolated vertices. By Lemma 2.5, there exists an acyclic digraph D such that $C(D) = C(D_1) \cup C(D_2) - \{i_2, i_3\} = (G - E(K)) \cup \{i_1\}$. Thus $k(G - E(K)) \leq 1$.

Case 3: The number of the components of $G - E(K)$ is 3.

Let G_1, G_2 and G_3 be the three components of $G - E(K)$. In this case, any two vertices of K are disconnected in $G - E(K)$, that is, there is no K-avoiding (v_i, v_j)-path for each distinct pair $i, j \in \{1, 2, 3\}$, and so no edge of K is on a hole in G. Therefore the two holes C_1 and C_2 of G remain

in $G - E(K)$. We consider the following two subcases:

Subcase 3-1: The two holes are contained in the same component of $G - E(K)$.

Without loss of generality, we may assume that G_1 and G_2 have no holes and G_3 contains the two holes. Then G_1 and G_2 are trees. Therefore there exist acyclic digraphs D_1 and D_2 such that $C(D_1) = G_1 \cup \{i_1\}$ and $C(D_2) = G_2 \cup \{i_2\}$, where i_1 and i_2 are new isolated vertices. Let x_1 and y_1 be two vertices which have no in-neighbors in D_1 and x_2 and y_2 be two vertices which have no in-neighbors in D_2. Since G_3 is connected and triangle-free and has exactly two holes, $k(G_3) = |E(G_3)| - |V(G_3)| + 2 = 3$. Then there exists an acyclic digraph D_3 such that $C(D_3) = G_3 \cup \{i_3, i_4, i_5\}$, where i_3, i_4, and i_5 are new isolated vertices. By Lemma 2.5, there exists an acyclic digraph D^* such that $C(D^*) = C(D_1) \cup C(D_2) - \{i_2\} = G_1 \cup G_2 \cup \{i_1\}$. Then, by Lemma 2.5 again, there exists an acyclic digraph D such that $C(D) = C(D^*) \cup C(D_3) - \{i_3, i_4, i_5\} = G_1 \cup G_2 \cup \{i_1\}$. Thus $k(G - E(K)) \leq 1$.

Subcase 3-2: The two holes are contained in different components of $G - E(K)$.

Without loss of generality, we may assume that G_1 have no holes and G_2 and G_3 contain exactly one hole. Then G_1 is a tree. Therefore there exists an acyclic digraphs D_1 such that $C(D_1) = G_1 \cup \{i_1\}$, where i_1 is a new isolated vertex. Let x_1 and y_1 be two vertices which have no in-neighbors in D_1. Since G_l is connected and triangle-free and has one hole, $k(G_l) = |E(G_l)| - |V(G_l)| + 2 = 2$ for $l = 2, 3$. Then there exist acyclic digraphs D_2 and D_3 such that $C(D_2) = G_2 \cup \{i_2, i_3\}$ and $C(D_3) = G_3 \cup \{i_4, i_5\}$, where i_2, i_3, i_4, and i_5 are new isolated vertices. Let x_2 and y_2 be two vertices which have no in-neighbors in D_2. By Lemma 2.5, there exists an acyclic digraph D^* such that $C(D^*) = C(D_1) \cup C(D_2) - \{i_2, i_3\} = G_1 \cup G_2 \cup \{i_1\}$. Then, by Lemma 2.5 again, there exists an acyclic digraph D such that $C(D) = C(D^*) \cup C(D_3) - \{i_4, i_5\} = (G - E(K)) \cup \{i_1\}$. Thus $k(G - E(K)) \leq 1$.

Hence, in any cases, we have $k(G - E(K)) \leq 1$. Let D' be an acyclic digraph such that $C(D') = (G - E(K)) \cup \{i_1\}$, where i_1 is a new isolated vertex. We define a digraph D by $V(D) = V(G) \cup \{i_1, i_2\}$ and $A(D) = A(D') \cup \{(v, i_2) \mid v \in K\}$, where i_2 is a new isolated vertex. Then D is acyclic and $C(D) = G \cup \{i_1, i_2\}$. Furthermore, all the vertices of K have i_2 as a common out-neighbor in D. Hence the lemma holds. $\square$

Now we will prove the main result.

Theorem 2.7. *Let G be a connected graph with exactly h holes. Suppose that the holes in G are pairwise edge-disjoint and that G has exactly one non-edge maximal clique. If the non-edge maximal clique K in G has size $h+1$, then $k(G) \leq 2$. In particular, there exists an acyclic digraph D such that $C(D) = G \cup \{i_1, i_2\}$ and all vertices of K have a common out-neighbor i_1 or i_2, where i_1 and i_2 are new isolated vertices.*

Proof. We prove the theorem by induction on the number of edge-disjoint holes. The basis step was already shown in the Lemma 2.6. Let $h \geq 2$. We assume that, for any graph G with exactly one maximal clique of size $h+1$ and exactly h edge-disjoint holes, there is an acyclic digraph D such that $C(D) = G \cup \{i_1, i_2\}$ and all vertices of K have a common out-neighbor in $\{i_1, i_2\}$. Now let G be a graph with just one maximal clique K of size $h+2$ and exactly $h+1$ edge-disjoint holes. We denote the vertices of K by $v_1, v_2, \ldots, v_{h+2}$ and the holes of G by $H_1, H_2, \ldots, H_{h+1}$. By Lemma 2.3, G contains a vertex v_i satisfying the condition (a) or (b). With out loss of generality, we may assume $v_i = v_1$.

First, suppose that v_1 satisfies the condition (a). By Lemma 2.4, $G - e$ has at most h edge-disjoint holes for an edge $e = uw \in E(H_i) \setminus E(K)$. Consider the graph $G' = G - \{v_1 v_j \mid v_j \in V(K) \setminus \{v_1\}\} - \{e\}$. Since v_1 satisfies (a), v_1 must belong to a component not containing holes or u or w in G' and G' has exactly two components. Let G_1 be the component containing v_1 and G_2 be the other components of G'. Since G_1 is a tree and the competition number of a tree is at most 1, there exists an acyclic digraph D_1 such that $C(D_1) = G_1 \cup \{i_1\}$ where i_1 is a new isolated vertex, and D_1 has at least two vertices , say x and y, of indegree 0. Since G_2 has a unique maximal clique of size $h+1$ and exactly h edge-disjoint holes, by the induction hypothesis, there exists an acyclic digraph D_2 such that $C(D_2) = G_2 \cup \{i_2, i_3\}$ where i_2 and i_3 are isolated vertices and all the vertices of $K - v_1$ has a common out-neighbor i_2 in D_2. By Lemma 2.5, there exists an acyclic digraph D^* such that $C(D^*) = C(D_1) \cup C(D_2) - \{i_3\} = G_1 \cup G_2 \cup \{i_1, i_2\}$. Moreover, all the vertices of $K - v_1$ has a common out-neighbor i_2 in D^*. Now we add arcs (v_1, i_2), (u, y), (w, y) to D^* to obtain a digraph D. It can easily be checked that D is acyclic and $C(D) = G \cup \{i_1, i_2\}$, and that all the vertices in K have a common out-neighbor i_2.

Second, suppose that v_1 satisfies the condition (b). Then v_1 is incident to an edge e shared by K and a hole H_j, and is not a vertex on any other hole. Without loss of generality, we may assume $H_j = H_1$. Then $G' := G - \{v_1 v_j \mid v_j \in V(K) \setminus \{v_1\}\}$ has a unique maximal clique $K - v_1$. By Lemma 2.4, G' has at most h edge-disjoint holes since we removed all the edges incident to v_1 in K. By the induction hypothesis, there exists an acyclic digraph D' such that $C(D') = G' \cup \{i_1, i_2\}$ where i_1 and i_2 are isolated vertices added and all the vertices of $K - v_1$ have a common out-neighbor i_1 in D'. Now, we define a digraph D by $V(D) = V(G) \cup \{i_1, i_2\}$ and $A(D) = A(D') \cup \{(v_1, i_1)\}$. Then it can easily be checked that D is acyclic and $C(D) = G \cup \{i_1, i_2\}$ and that all the vertices in K have a common out-neighbor i_1. □

Theorem 2.7 can be generalized as follows:

Theorem 2.8. *Let G be a connected graph with exactly h holes. Suppose that the holes in G are pairwise edge-disjoint and that G has exactly one non-edge maximal clique. If the size ω of the non-edge maximal clique in G satisfies $3 \leq \omega \leq h+1$, then $k(G) \leq h - \omega + 3$.*

Proof. Let G be a connected graph with exactly $h \geq 2$ edge-disjoint holes $H_1, H_2, \dots, H_h$ and exactly one non-edge maximal clique K of size ω, $3 \leq \omega \leq h+1$. Since the bound holds when $\omega = h+1$ by Theorem 2.7, we deal with the case $\omega < h+1$. We take an edge $e_j \in E(H_j) \setminus E(K)$ for each $j = 1, 2, ..., h$. Let F be the set of such edges and F' be a subset of F with $h+1-\omega$ elements. Let $G' = G - F'$. Then G' still has a unique maximal clique K. Moreover, since $e_j \in E(H_j) \setminus E(K)$ for each j, G' has exactly $\omega - 1$ edge-disjoint holes by Lemma 2.4. Thus, $k(G') \leq 2$ by Theorem 2.7. Then there exists an acyclic digraph D' such that $C(D') = G' \cup I_2$. Now we add vertices $i_1, \dots, i_{h+1-\omega}$ and arcs from the ends of e_j to i_j for $j = 1, \dots, h+1-\omega$ to D' to obtain D. Then it is easy to check that D is acyclic and $C(D) = G \cup I_{2+(h+1-\omega)}$. Hence $k(G) \leq h - \omega + 3$. □

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